T.D.C. II

Part – 4

Some Examples on basis of Taylor's formula

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Example ①:- Obtain Taylor's formula for the function e^{x+y} at $(0,0)$ for $n=3$.

Solution: For $n=3$, Taylor's formula is given by

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f(0,0) + \frac{1}{\lfloor 2}\left(x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2}\right)f(0,0) + \frac{1}{\lfloor 3}\left(x^3\frac{\partial^3}{\partial x^3} + 3x^2y\frac{\partial^3}{\partial x^2\partial y} + 3xy^2\frac{\partial^3}{\partial x\partial y^2} + y^3\frac{\partial^3}{\partial y^3}\right)f(\theta x, \theta y),$$

where $0<\theta<1$ — ①

We have $f(x,y) = e^{x+y}$

$\Rightarrow f(0,0) = e^{0+0} = e^0 = 1$

$$\left.\begin{aligned} &\frac{\partial f}{\partial x} = e^{x+y};\quad \frac{\partial f}{\partial y} = e^{x+y};\quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y^2} = e^{x+y} \\ &\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial x^2\partial y} = \frac{\partial^3 f}{\partial x\partial y^2} = \frac{\partial^3 f}{\partial y^3} = e^{x+y} \end{aligned}\right\} \text{—(ii)}$$

So at $(0,0)$; we have from (ii), $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 1$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y^2} = 1$

and substituting θx for x and θy for y; we obtained

$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial x^2\partial y} = \frac{\partial^3 f}{\partial x\partial y^2} = \frac{\partial^3 f}{\partial y^3} = e^{\theta(x+y)}$ —(iii)

Using eqn (iii) in eqn (i), we obtained.

$$e^{x+y} = 1 + (x+y) + \frac{1}{\lfloor 2}(x^2+2xy+y^2) + \frac{1}{\lfloor 3}(x^3+3x^2y+3xy^2+y^3)e^{\theta(x+y)}$$

$$e^{x+y} = 1 + (x+y) + \frac{1}{\lfloor 2}(x+y)^2 + \frac{1}{\lfloor 3}(x+y)^3 e^{\theta(x+y)}$$ is required formula

Exercise:- Prove by Taylor's formula

$$\sin x \sin y = xy - \frac{1}{6}\{(x^3+3xy^2)\cos\theta x \sin\theta y + (y^3+3x^2y)\sin\theta x\cos\theta y\},$$

where $0<\theta<1$.

Hint: For $n=3$.

Example ②:- Expand $\sin xy$ in powers of $(x-1)$ and $(y-\frac{\pi}{2})$ upto second degree terms.

Solution:- We have, given that

$$f(x,y) = \sin xy \quad \text{——— (i)}$$

$$\Rightarrow f(1,\tfrac{\pi}{2}) = \sin 1.\tfrac{\pi}{2} = \sin\tfrac{\pi}{2} = 1$$

And $\frac{\partial f}{\partial x} = y\cos xy$, $\frac{\partial f}{\partial y} = x\cos xy$

$$\frac{\partial^2 f}{\partial x^2} = -y^2\sin xy \; ; \; \frac{\partial^2 f}{\partial x\partial y} = \cos xy - xy\sin xy \quad \text{——— (ii)}$$

$$\frac{\partial^2 f}{\partial y^2} = -x^2\sin xy$$

At the point $(1,\frac{\pi}{2})$; eqn (ii) gives

$$\frac{\partial f}{\partial x} = 0; \; \frac{\partial f}{\partial y} = 0 \; ; \; \frac{\partial^2 f}{\partial x^2} = -\frac{\pi^2}{4} \; ; \; \frac{\partial^2 f}{\partial x\partial y} = \frac{\pi}{2}; \; \frac{\partial^2 f}{\partial y^2} = -1 \quad \text{(iii)}$$

By Taylor's theorem; we know that

$$f(x,y) = f(1,\tfrac{\pi}{2}) + \left[(x-1)\frac{\partial}{\partial x} + (y-\tfrac{\pi}{2})\frac{\partial}{\partial y}\right] f(1,\tfrac{\pi}{2})$$
$$+ \frac{1}{\lfloor 2}\left\{(x-1)^2\frac{\partial^2}{\partial x^2} + 2(x-1)(y-\tfrac{\pi}{2})\frac{\partial^2}{\partial x\partial y} + (y-\tfrac{\pi}{2})^2\frac{\partial^2}{\partial y^2}\right\} f(1,\tfrac{\pi}{2}) \quad \text{——— (iv)}$$

In above eqn using $f(1,\frac{\pi}{2}) = 1$ and eqn (iii); we have

$$\sin xy = 1 + \left[(x-1).0 + (y-\tfrac{\pi}{2}).0\right] + \frac{1}{2}\left\{(x-1)^2\left(-\frac{\pi^2}{4}\right) + 2(x-1)(y-\tfrac{\pi}{2})\cdot\frac{\pi}{2} + (y-\tfrac{\pi}{2})^2(-1)\right\}$$

$$\Rightarrow \sin xy = 1 - \frac{\pi^2}{8}(x-1)^2 + (x-1)(y-\tfrac{\pi}{2})\frac{\pi}{2} - \frac{1}{2}(y-\tfrac{\pi}{2})^2$$

which is required solution.

Exercise:- Expand $f(x,y) = \log(x+e^y)$ by Taylor's series in powers of $(x-1)$ and y such that it includes all terms upto second degree.